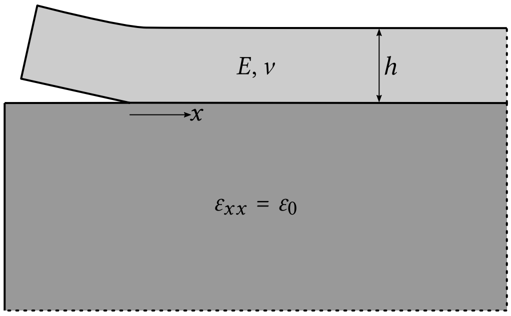
Exercise 13: fracture

22.01.2024 - 26.01.2024

Question 1 . . . .

A thick substrate of width w is coated with a thin film. The material of the film has Young's modulus E and Poisson ratio ν. There is a crack of length a between substrate and film. Now the substrate is subjected to a tensile strain ε_0, which is therefore also imposed on the film. Calculate the energy release rate $\mathcal{G}$!



Solution: We will assume that the plate is sufficiently wide so that coating is approximately in a state of plane strain. Moreover, the coating can relax along the direction perpendicular to the plate, i.e. the normal stress in this direction vanishes. Therefore, the coating is in a state of pure tensile stress for $x \gg h$. The modulus for pure tensile stress in plane strain is $E/(1-\nu^2)$, i.e.

$$\sigma = \frac{E}{1-\nu^2}\varepsilon_0.$$

The internal energy density is $\sigma\varepsilon_0/2$. When the crack advances by a short distance da, then the film contracts again over this distance. The change in energy is

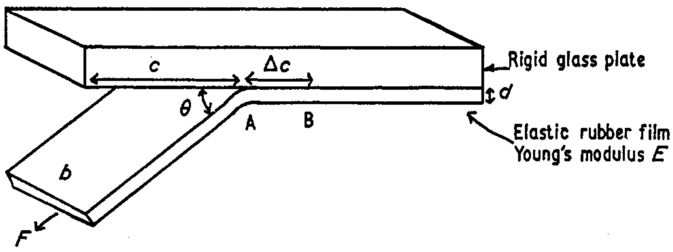
$$d\Pi = -\frac{1}{2}\sigma\varepsilon_0 whda = -\sigma^2\frac{1-\nu^2}{2E}Bhda.$$

The energy release rate is therefore

$$\mathcal{G} = \frac{1-\nu^2}{2E}\sigma^2 h.$$

Question 2 . . . .

An elastic film of thickness d and Young's modulus E is peeled at an angle θ from a rigid substrate and a constant force F. During peeling the loose end of the film is stretched and stores elastic energy. This elastic energy is available for the creation of a new surface. Remember that the Griffith criterion for fracture is the hypothesis, that the new surface is created if the released elastic energy is larger than the energy required for making the new surface. We now consider the energy balance during release of the rubber film between points A and B.



(a) Assume that $w = 2\gamma$ is the work of adhesion, i.e. the magnitude of the energy per surface area that is required to separate the film from the substrate. What is the energy E_{surf} required to advance the film by Δc?

(b) Once released, the section A-B of the film is elongated with respect to its original length Δc. What is the elongation $x = \varepsilon\Delta c$ (where ε is the strain) of that section of the film under the action of the force F?

(c) Compute the energy required for the extension of the section A-B of the film by integrating the elastic restoring force $f(x)$ over the elongation of the film, i.e. $E_{\text{el}} = \int_0^{\varepsilon\Delta c} dx\, f(x)$. Write the final expression in terms of F.

(d) Finally, the loose end of the film (to the left of point A) gains potential energy because it moves parallel to the force F. By what distance Δx does the loose end move parallel to F when the film section A-B is released from the interface? (There is a purely geometric contribution to this distance and a contribution that comes from streching the section A-B.)

(e) Compute the potential energy by integrating the force F over this distance, i.e. $E_{\text{pot}} = \int_0^{\Delta x} dx\, F$.

(f) Write down the total energy balance for this process. For this you need to consider which of the above processes release and which cost energy to determine their respective sign.

(g) What is the critical force F required to peel the rubber film at an angle θ from the glass substrate? Is there an angle at which peeling is no longer possible?

Solution: Reference: K. Kendall, *Thin-film peeling - the elastic term*, J. Phys. D 8, 1449 (1975)

(a) The are that is release from the substrate is $b\Delta c$, the surface energy (cost) is hence $E_{\text{surf}} = wb\Delta c$.

(b) The stress inside the film is $\sigma = F/bd$. This leads to a strain of $\varepsilon = \sigma/E = F/bdE$. The elongation is then given by $x = \varepsilon\Delta c = F\Delta c/bdE$.

(c) Given the stress in the film $\sigma = E\varepsilon$ and the strain at elongation x of $\varepsilon = x/\Delta c$, the elastic restoring force is given by $f(x) = \sigma(x)bd = bdEx/\Delta c$. We now integrate

$$E_{\text{el}} = \int_0^{\varepsilon\Delta c} dx\, \frac{bdE}{\Delta c}x = \frac{1}{2}\frac{bdE}{\Delta c}\varepsilon^2\Delta c^2 = \frac{\Delta c}{2bdE}F^2 \tag{1}$$

(d) The elongation due to stretching was computed in the part (b). It is given by

$$\Delta x_s = \varepsilon\Delta c = \frac{F\Delta c}{bdE}. \tag{2}$$

The geometric contribution is given by

$$\Delta x_g = \Delta c - \Delta c\cos\theta = (1-\cos\theta)\Delta c. \tag{3}$$

The strip to the left of point A hence moves a distance

$$\Delta x = \frac{F\Delta c}{bdE} + (1-\cos\theta)\Delta c \tag{4}$$

in the direction of the force F.

(e) Since the force F is constant, the energy is simply given by $E_{\text{pot}} = F\Delta x$.

(f) E_{surf} is an energy cost (negative sign), E_{el} is an energy cost (negative sign), but E_{pot} is the work performed on the system (postive sign). The total energy balance is hence

$$\Delta E = E_{\text{pot}} - E_{\text{el}} - E_{\text{surf}} = \frac{\Delta c}{2bdE}F^2 + (1-\cos\theta)\Delta cF - wb\Delta c \tag{5}$$

The strip peels if $\Delta E > 0$. It is useful to normalize ΔE by the area $b\Delta c$ of the peeling from. This gives

$$\frac{\Delta E}{b\Delta c} = \frac{1}{2dE}\left(\frac{F}{b}\right)^2 + (1-\cos\theta)\frac{F}{b} - w \tag{6}$$

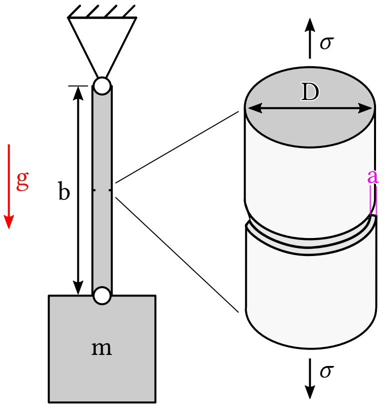
(g) Solving for F yields

$$\frac{F}{b} = -dE(1-\cos\theta) \pm \sqrt{d^2E^2(1-\cos\theta)^2 + 2dEw} \tag{7}$$

where only the + sign is relevant (because the other sign leads to a negative force). Peeling is always possible since $F > 0$ for any angle θ.

Question 3 . . . .

A bar of length b and negligible mass is fixed at the ceiling and carries a much heavier body of mass m. Gravity acts downwards and the gravitational acceleration is g. The bar has a cylindrical cross-section with a diameter D. It is made of a material with Young's modulus E, Poisson ratio ν and critical stress intensity factor K_{Ic}. In the middle of the bar there is a circumferential notch of depth $a = D/10$. Determine the maximum mass m that the bar can sustain without breaking! Use the approximation for the stress intensity factor K_I that is shown in the figure!



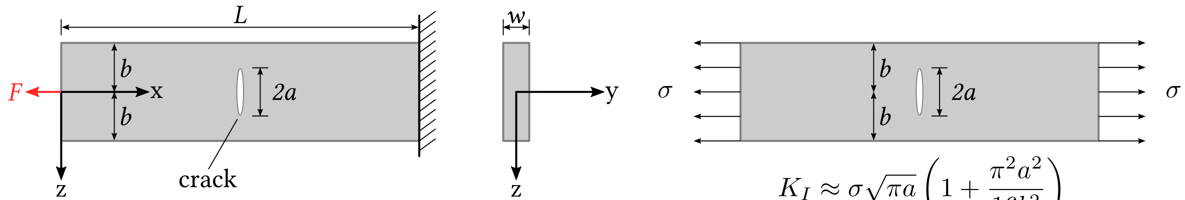
$$K_I = \sigma\,(\pi a)^{1/2}\,\frac{1 - 3\,(a/D)}{\left[1 - 2\,(a/D)^2\right]^{3/2}}$$

Solution: The normal force acting on the bar is $F = mg$. This leads to a stress $\sigma F/A = 4F/\pi D^2 = 4mg/\pi D^2$ inside the bar. The bar breaks when $K_I > K_{Ic}$, i.e. when

$$\begin{aligned} K_{Ic} &< \frac{4mg}{\pi D^2}\,(\pi a)^{\frac{1}{2}}\,\frac{1-3(1/10)}{[1-2(1/10)^2]^{3/2}} \\ &= \frac{4mg}{\pi D^{3/2}}\,(\pi a/D)^{1/2}\,\frac{0.7}{0.98^{3/2}} \\ &= \frac{2.8}{(10\cdot 0.98^3\cdot\pi)^{1/2}}\,\frac{mg}{D^{3/2}} \\ &\approx 0.515\frac{mg}{D^{3/2}} \end{aligned}$$

Question 4 . . . .

A cantilever beam of length L, thickness $2b$ and width w is subjected to a normal force F at the end. The beam is made of a brittle material with critical stress intensity factor K_{Ic}, and contains a crack of length $2a$ at the center. What is the maximum value of F at which the beam breaks? The stress intensity factor K_I for a plate with center crack of length $2a$ under constant uniaxial tension is given in the figure.



$$K_I \approx \sigma\sqrt{\pi a}\left(1 + \frac{\pi^2 a^2}{16b^2}\right)$$

Solution: The normal stress in the beam is

$$\sigma = \frac{F}{2wb},$$

hence the stress intensity K_I is

$$K_I = \frac{F}{2wb}\sqrt{\pi a}\left(1 + \frac{\pi^2 a^2}{16b^2}\right).$$

The beam breaks if $K_I > K_{Ic}$, therefore the maximum value of F is

$$F = \frac{2wb}{\sqrt{\pi a}} \left(1 + \frac{\pi^2 a^2}{16b^2}\right)^{-1} K_{Ic}.$$