Exercise 12: Yield and fracture
Jan. 31, 2022 - Feb. 04, 2022

Question 1 . . . .
Reference: Rösler, Harders, Bäker, Mechanisches Verhalten der Werkstoffe, 2nd ed, Teubner, p. 412

A component made from a polycrystalline Aluminum alloy has the yield strength of $200\,\mathrm{MPa}$ and is subjected to plane stress with

$$\sigma_{xx} = \sigma_{yy} = 155\,\mathrm{MPa}, \quad \tau_{xy} = 55\,\mathrm{MPa}.$$

(a) Write the deviatoric stress!

(b) Calculate the principal stresses!

(c) Evaluate both Tresca's and von Mises' criterion to determine whether the material will yield!

Solution:

(a)

$$\sigma' = \sigma - \frac{1}{3}\operatorname{Tr}\sigma\mathbf{1} = \begin{bmatrix} 155\,\mathrm{MPa} & 55\,\mathrm{MPa} & 0 \\ 55\,\mathrm{MPa} & 155\,\mathrm{MPa} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{310}{3}\,\mathrm{MPa}\mathbf{1} = \begin{bmatrix} \frac{155}{3}\,\mathrm{MPa} & 55\,\mathrm{MPa} & 0 \\ 55\,\mathrm{MPa} & \frac{155}{3}\,\mathrm{MPa} & 0 \\ 0 & 0 & -\frac{310}{3}\,\mathrm{MPa} \end{bmatrix}$$

(b)

$$\sigma_I = 210\,\mathrm{MPa},\ \sigma_{II} = 100\,\mathrm{MPa},\ \sigma_{III} = 0\,\mathrm{MPa}$$

(c) The two criteria contradict in this case,

$$\sigma_{\text{Tresca}} = \sigma_I - \sigma_{III} = 210\,\mathrm{MPa} > 200\,\mathrm{MPa} \implies \text{the material will yield},$$

$$\sigma_{\text{Mises}} = \sqrt{\frac{1}{2}\left[(\sigma_I - \sigma_{II})^2 + (\sigma_{II} - \sigma_{III})^2 + (\sigma_I - \sigma_{III})^2\right]} = 181.93\,\mathrm{MPa} \implies \text{the material will not yield.}$$

Both criteria are empirical, hence no decision can be made.

Question 2 . . . .
Reference: Sun, Fracture Mechanics, 1st ed, Academic Press (p. 72)

Consider the following Airy stress function,

$$\phi = Ay^2,$$

where A is a constant. Compute the stress components and the displacements, assuming zero rigid body rotation.

Solution: The stresses are

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = 2A,$$

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = 0,$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = 0.$$

Using this result in Hooke's law, we get

$$\begin{aligned} 2A &= \lambda\left(\varepsilon_{xx}+\varepsilon_{yy}\right)+2\mu\varepsilon_{xx}, \\ 0 &= \lambda\left(\varepsilon_{xx}+\varepsilon_{yy}\right)+2\mu\varepsilon_{yy}, \\ 0 &= 2\mu\varepsilon_{xy}, \end{aligned}$$

which can be solved for the components of strain,

$$\begin{aligned} \varepsilon_{xx} &= \frac{(2\mu+\lambda)}{(\mu+\lambda)}\frac{A}{2\mu}, \\ \varepsilon_{yy} &= -\frac{\lambda}{(\mu+\lambda)}\frac{A}{2\mu}. \end{aligned}$$

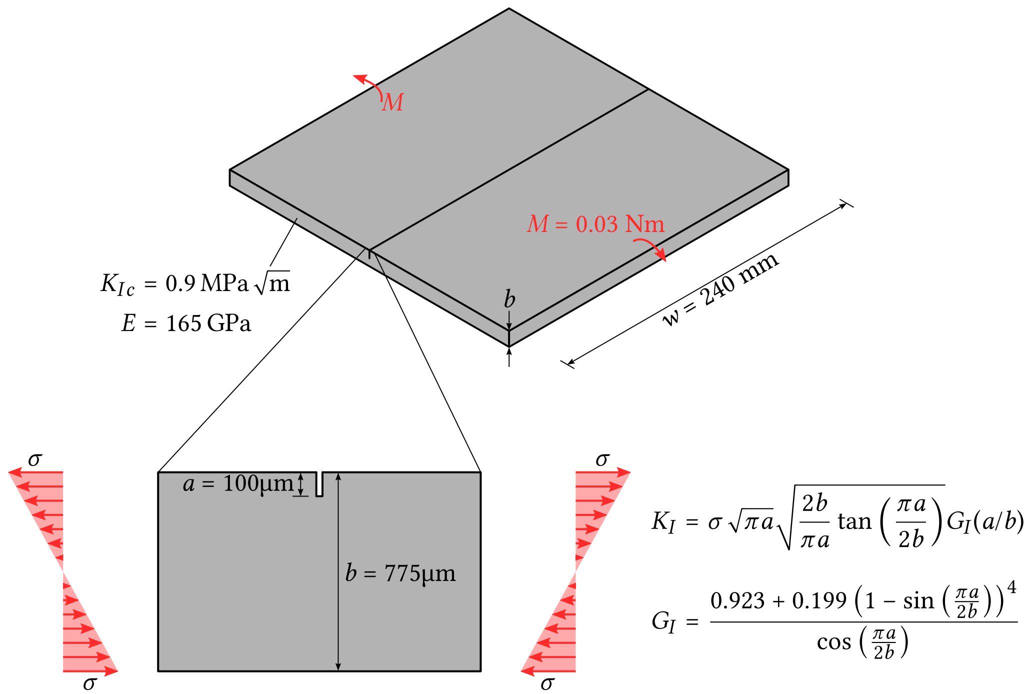
Finally, the displacements can be obtained by integrating strain,

$$\begin{aligned} \frac{\partial u_x}{\partial x} &= \varepsilon_{xx} \implies u_x = \frac{(2\mu+\lambda)}{(\mu+\lambda)}\frac{A}{2\mu}x + C_1(y) + C_2, \\ \frac{\partial u_y}{\partial y} &= \varepsilon_{yy} \implies u_y = -\frac{\lambda}{(\mu+\lambda)}\frac{A}{2\mu}y + C_3(x) + C_4, \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}\right) &= \varepsilon_{xy} = 0. \end{aligned}$$

We were given the information that rigid body rotation is zero, hence $(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}) = 0$. In conjunction with the last equations, this means $\frac{\partial u_x}{\partial y} = \frac{\partial u_y}{\partial x}$, and hence $C_1(y) = C_3(x) = 0$. Therefore, the displacements are determined up to a rigid body displacement given by C_2 and C_4.

Question 3 . . . .

A Si plate has a thickness of $775\,\mu\mathrm{m}$ and a width of $240\,\mathrm{mm}$. It has a rectangular cross-section and contains a $100\,\mu\mathrm{m}$ deep notch. The plate is subjected to pure bending with a moment of $M = 0.03\,\mathrm{N\,m}$. Will it break? K_I for this load case is given in the figure. The material has fracture toughness $K_{1c} = 0.9\,\mathrm{MPa}\sqrt{\mathrm{m}}$ and a Youngs modulus of $E = 165\,\mathrm{GPa}$.



Solution: We first need to find the stress σ, i.e. the maximum value of normal stress at the top of the plate. The relation between the bending moment and the normal stress was discussed in chapter 11 of the lecture. Let us denote the bending axis by y and the direction perpendicular to the plate by z. The normal stress as a function of z is then

$$\sigma_{xx}(x, z) = \frac{M_y}{I_{yy}} z, \tag{1}$$

where I_{yy} is the area moment. The plate has a simple rectangular cross-section, hence $I_{yy} = b^3 w/12$ (see exercise 9). Thus, the stress at $z = b/2$ is $\sigma = 6M_y b/(b^3 w) = 1.249\,\mathrm{MPa}$. With $a/b = 0.129$, we get for the geometry factor $G_I = 1.025$. Inserting this value and σ into the formula for K_I, we get $K_I = 0.023\,\mathrm{MPa}\sqrt{\mathrm{m}}$. This value is below the critical value of $K_{Ic} = 0.9\,\mathrm{MPa}\sqrt{\mathrm{m}}$, therefore we predict that the plate will not break.