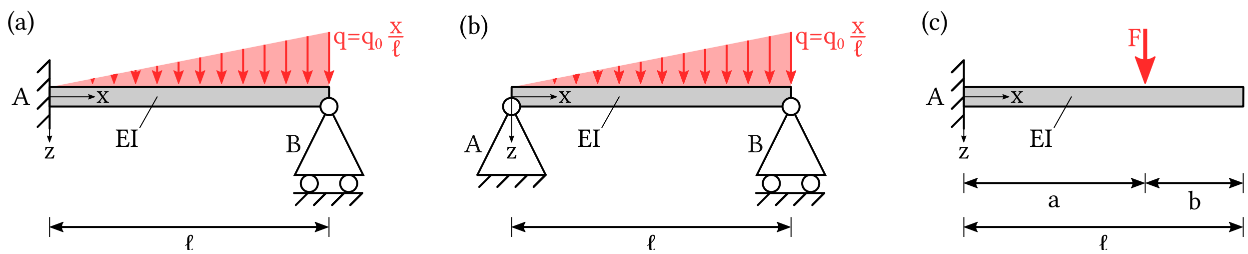
Exercise 10: Bending and buckling

18.12.2023 - 22.12.2023

Question 1 . . . .

Determine if the following structures are statically determinate! Calculate the deflection and the reaction forces and moments at the supports!



Solution: Recall that a structure is statically determinate if

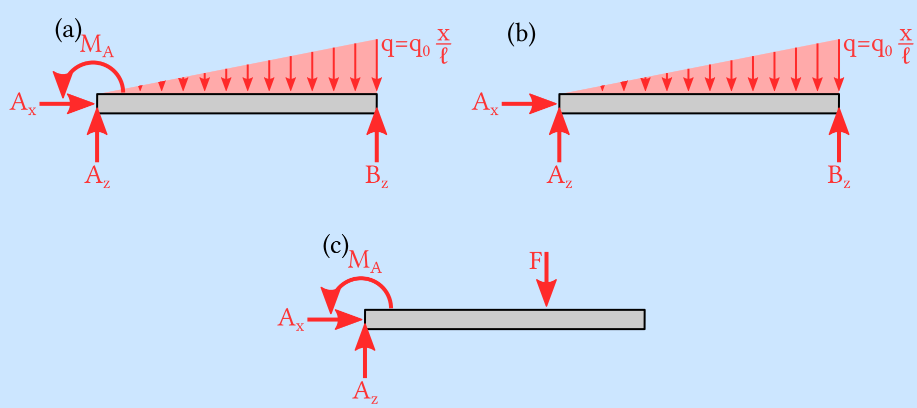
$$3n - (r + v) = 0,$$

where n is the number of bodies, r the number of reaction forces or moments of the supports, and v the number of forces or moments transmitted at links.

(a) $n = 1$, $r = 4$, $v = 0 \rightarrow$ indeterminate / overconstrained

(b) $n = 1$, $r = 3$, $v = 0 \rightarrow$ determinate

(c) $n = 1$, $r = 3$, $v = 0 \rightarrow$ determinate



In order to calculate the z-deflection $w(x)$ of the beam we have to integrate the Euler-Bernoulli equation. However, there are two possible starting points. Either we integrate $EIw''''(x)$ four times, or we first determine the bending moment as a function of position ($M(x)$) and then integrate $EIw''(x) = -M(x)$ two times. In both cases we need to make use of the boundary conditions in order to determine the constants of integration. However, in the second case, there will be fewer constants of integration.

Since problem (a) is indeterminate we cannot immediately calculate $M(x)$ and therefore need to follow the first approach. Note that problems (a) and (b) have the same geometry, load and boundary condition on the right hand side. Hence, (b) can be solved quickly by recycling the solution of (a).

(a)

$$
\begin{aligned}
EIw''''(x) &= q_0 \frac{x}{l} \\
EIw'''(x) &= \frac{1}{2} q_0 \frac{x^2}{l} + C_1 \\
EIw''(x) &= \frac{1}{6} q_0 \frac{x^3}{l} + C_1 x + C_2 \\
EIw'(x) &= \frac{1}{24} q_0 \frac{x^4}{l} + \frac{1}{2} C_1 x^2 + C_2 x + C_3 \\
EIw(x) &= \frac{1}{120} q_0 \frac{x^5}{l} + \frac{1}{6} C_1 x^3 + \frac{1}{2} C_2 x^2 + C_3 x + C_4
\end{aligned}
$$

Now we apply boundary conditions:

$$
\begin{aligned}
w(x=0) = 0 &\implies C_4 = 0, \\
w'(x=0) = 0 &\implies C_3 = 0, \\
\left.\begin{aligned} w''(x=l) = 0 \quad \text{(zero moment)} \\ w(x=l) = 0 \end{aligned}\right\} &\implies C_1 = -\frac{9}{40} q_0 l, \quad C_2 = \frac{7}{120} q_0 l^2.
\end{aligned}
$$

The solution for the displacements can thus be written as

$$
w(x) = \frac{1}{EI} \left(\frac{1}{120} q_0 \frac{x^5}{l} - \frac{3}{80} q_0 l x^3 + \frac{7}{240} q_0 l^3 x \right).
$$

The reaction forces and moments can be obtained by evaluating the appropriate derivatives of $w(x)$ at the location of the bearings. To get the correct sign, draw the reaction forces and moments with an arbitrary sense. Then imagine a cut and require that the bearing force/moment and the reaction force/moment sum to zero.

$$
\begin{aligned}
A_z &= Q(x=0) = -EIw'''(x=0) = -C_1 = \frac{9}{40} q_0 l, \\
B_z &= -Q(x=l) = EIw'''(x=l) = \frac{11}{40} q_0 l, \\
M_A &= -M(x=0) = EIw''(x=0) = C_2 = \frac{7}{120} q_0 l^2, \\
A_x &= 0 \quad \text{(equilibrium)}
\end{aligned}
$$

We could check the solution for A_z, B_z and M_A by checking equilibrium of the whole structure.

(b) The line load is the same in (a), therefore the leading term of $w(x)$ is $q_0 x^5/120l$, as before. The boundary conditions are

$$
w(x=0) = 0 \implies C_4 = 0, \quad w''(x=0) = 0 \implies C_2 = 0, \quad w''(x=l) = 0 \quad \text{(zero moment)} \implies C_1 = -\frac{1}{6} q_0 l
$$

$$
w(x=l) = 0 \implies C_3 = \frac{7}{360} q_0 l^3,
$$

hence the solution for the displacements is

$$
w(x) = \frac{1}{EI} \left(\frac{1}{120} q_0 \frac{x^5}{l} - \frac{1}{36} q_0 l x^3 + \frac{7}{360} q_0 l^3 x \right).
$$

The reactions forces are

$$
\begin{aligned}
A_z &= Q(x=0) = -EIw'''(x=0) = -C_1 = \frac{1}{6} q_0 l, \\
B_z &= -Q(x=l) = EIw'''(x=l) = \frac{1}{3} q_0 l.
\end{aligned}
$$

The same solution should be obtained by consideration of equilibrium of the whole structure.

(c) The problem is statically determinate, hence the reactions at the support can be obtained by requiring equilibrium of the whole structure,

$$A_x = 0,$$
$$A_z = F,$$
$$M_A = Fa.$$

We divide the structure into two sectors, $0 \leq x \leq a$ (sector 1) and $a \leq x \leq l$ (sector 2). The internal moment is

$$M^{(1)}(x) = F\left(x - a\right),$$
$$M^{(2)}(x) = 0.$$

The deflection is obtained by integrating the second derivative of $w''(x)$.

$$\begin{aligned}
EI{w^{(1)}}''(x) &= F\left(x-a\right),\\
EI{w^{(1)}}'(x) &= -\frac{1}{2}Fx^2 + Fax + C_1,\\
EIw^{(1)}(x) &= -\frac{1}{6}Fx^3 + \frac{1}{2}Fax^2 + C_1x + C_2,\\
EI{w^{(2)}}''(x) &= 0,\\
EI{w^{(2)}}'(x) &= C_3,\\
EIw^{(2)}(x) &= C_3x + C_4.
\end{aligned}$$

Consideration of the boundary conditions gives the solution for the constants of integration,

$$\begin{aligned}
{w^{(1)}}'(x=0) = 0 &\implies C_1 = 0,\\
w^{(1)}(x=0) = 0 &\implies C_2 = 0,\\
{w^{(1)}}'(x=a) = {w^{(2)}}'(x=a) &\implies C_3 = \frac{1}{2}Fa^2,\\
w^{(1)}(x=a) = w^{(2)}(x=a) &\implies C_4 = -\frac{1}{6}Fa^3.
\end{aligned}$$

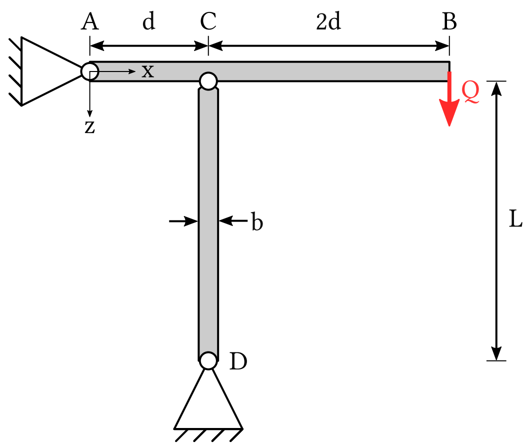
The solution for the deflection is therefore

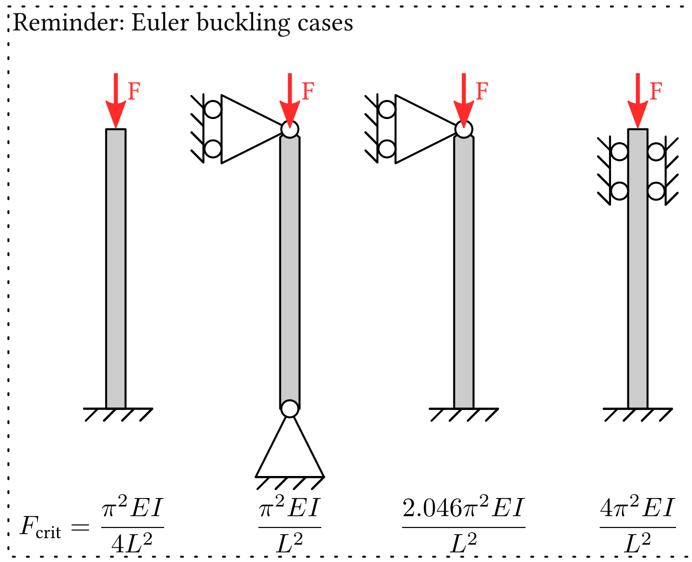
$$w^{(1)}(x) = \frac{1}{EI}\left[-\frac{1}{6}Fx^3 + \frac{1}{2}Fax^2\right],$$
$$w^{(2)}(x) = \frac{1}{EI}\left[\frac{1}{2}Fa^2x - \frac{1}{6}Fa^3\right].$$

Question 2 . . . .

Reference: Gere and Timoshenko, Mechanics of Materials, 4th ed., PWS Publishing Company (p. 789)

A horizontal beam AB is supported by a pinned-end column CD, as shown in the figure. The column is a solid steel bar (Young's modulus $E = 200\,\mathrm{GPa}$) of square cross-section having length $L = 1.8\,\mathrm{m}$ and side dimensions $b = 50\,\mathrm{mm}$. For safety reasons, the normal force in column CD should not exceed *half* the critical buckling force F_{crit}. Determine the maximum allowable force Q!





Solution: We first need to compute the normal force F in column CD. We cut the two members at point C. The equilibrium conditions yield

$$\begin{aligned} &\circlearrowleft C \quad && -A_z d - 2Qd = 0 \implies A_z = -2Q \\ &\uparrow && A_z + F - Q = 0 \quad \implies F = Q - A_z = 3Q \end{aligned}$$

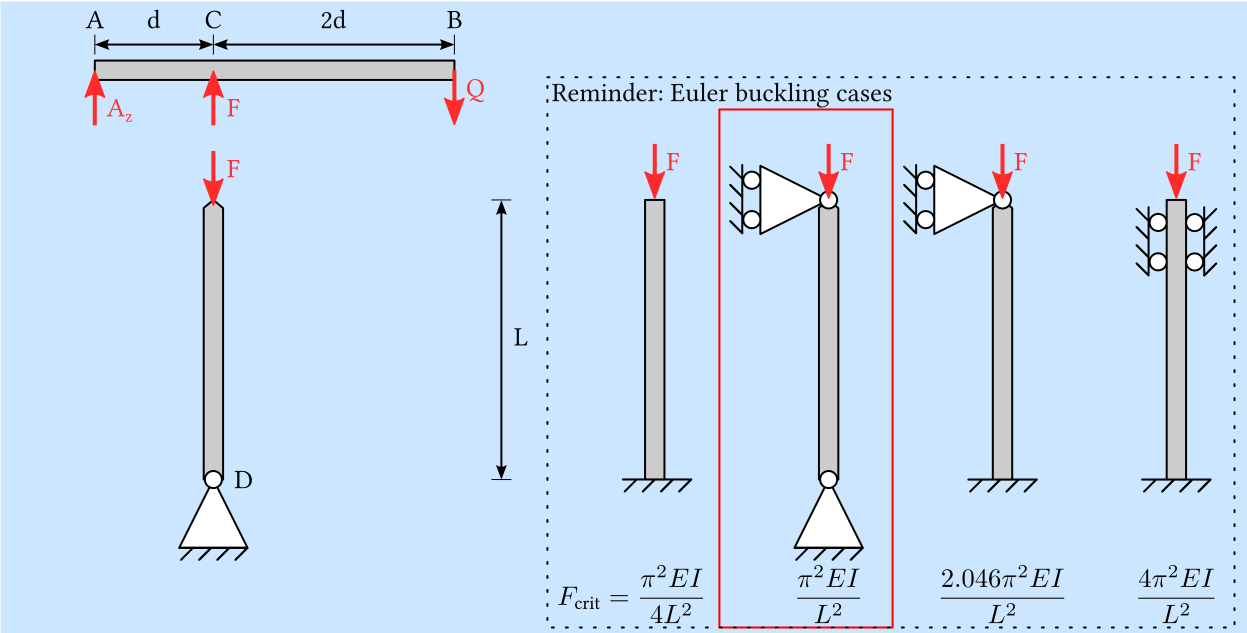
The second Euler buckling case is the relevant case. F must not exceed half the critical load, so the requirement is

$$F \leq \frac{1}{2}\frac{\pi^2 EI}{L^2},$$
$$\leftrightarrow Q \leq \frac{1}{6}\frac{\pi^2 EI}{L^2}.$$

In the previous exercise, we computed the second moment of area for a rectangular cross-section with side lengths b and h as $I = bh^3/12$. For a square $h = b$, hence $I = b^4/12$, and therefore

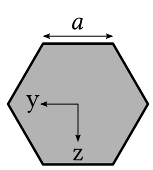
$$Q \leq \frac{1}{72}\frac{\pi^2 E b^4}{L^2}.$$

By inserting the known material parameters and dimensions, we find $Q \leq 52.9\,\text{kN}$.



Question 3 ....

Calculate the second moment of area I_y for a regular hexagon:

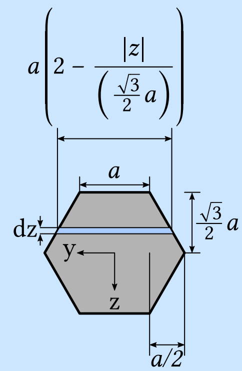


Solution: Start from the definition of the second moment of inertia I_y

$$I_y = \int z^2 dA$$

The infinitesimal area dA can be computed like sketched in the figure. Recognize that a hexagon can be constructed from six equilateral triangles of edge length a, which have a height of $h = \frac{\sqrt{3}}{2}a$. Thus the coordinate z is from the interval $[-h, +h]$. The length in y-direction is a linear function of $|z|$ and goes from $\Delta y = 2a$ at $z = 0$ to $\Delta y = a$ at $z = \pm h$. We find $Deltay(z) = a\left(2 - \frac{|z|}{\left(\frac{\sqrt{3}}{2}a\right)}\right)$ and $dA = \Delta y(z)dz$. The integral becomes

$$I_y = \int_{-\sqrt{3}/2a}^{\sqrt{3}/2a} z^2 a \left(2 - \frac{|z|}{\left(\frac{\sqrt{3}}{2}a\right)}\right) dz = \frac{5\sqrt{3}a^4}{16}$$



$a\left(2-\frac{|z|}{\left(\frac{\sqrt{3}}{2}a\right)}\right)$
a
dz
$\frac{\sqrt{3}}{2}a$
y
z
$a/2$