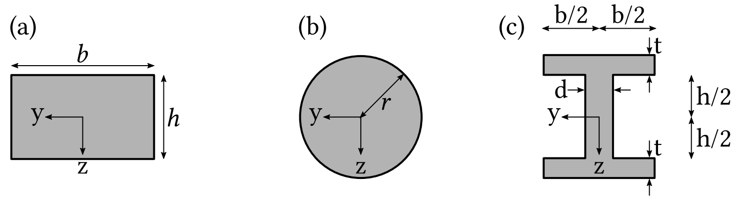
Exercise 9: Bending
11.12.2023 - 15.12.2023

Question 1 . . . .

Calculate the second moment of area I_y for the following profiles:



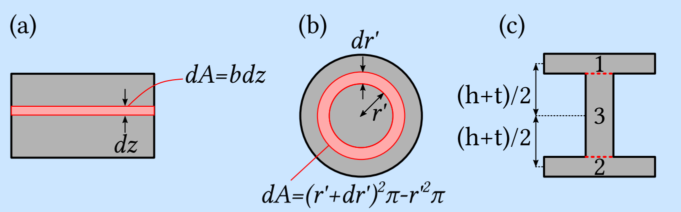
Hints:

- For the solution of (b) it is useful to consider the polar moment $I_r = \int r^2 dA = \int (y^2 + z^2) dA$. From the symmetry of the problem it follows that $I_y = I_z$.
- For the solution of (c) you can use the result from (a). Decompose the cross-section into rectangles and sum their respective I_y to get I_y of the whole cross-section. You will need the parallel axis theorem (Huygens-Steiner theorem), which says that the moment $I_{\bar{y}}$ for bending about an axis $\bar{y}$ that is parallel to y but separated by a distance l is $I_{\bar{y}} = I_y + l^2 A$, where A is the area.

Solution: We need to compute the integral

$$I_y = \int z^2 dA$$

for the depicted cross-sections. We will need to specify what dA is in each case.



(a) The area element dA is a strip of width b and height dz.

$$dA = bdz$$
$$\rightarrow I_y = \int_{-h/2}^{h/2} z^2 bdz = \frac{1}{12} h^3 b$$

(b) We'll start by calculating the polar moment $I_r = \int r'^2 dA$. Here, dA is a ring of thickness dr'. One can either reason from the geometrical considerations how dA looks like

$$dA = (r' + dr')^2 \pi - r'^2 \pi = 2r'dr'\pi + dr'^2 \pi \approx 2rdr\pi$$
$$\rightarrow I_r = \int_0^r 2\pi r'^3 dr' = \frac{\pi}{2} r^4$$

or one could remember cylindrical coordinates to find

$$dA = rdrd\varphi$$
$$\rightarrow I_r = \int_0^{2\pi} \int_0^r r'^3 dr' d\varphi = \frac{\pi}{2} r^4$$

Now $I_r = \int r^2 dA = \int (y^2 + z^2) dA = I_y + I_z$. The cross-section is symmetric with respect to bending about y and z. Therefore $I_y = I_z = {}^1\!/_2 I_r = \frac{\pi}{4} r^4$.

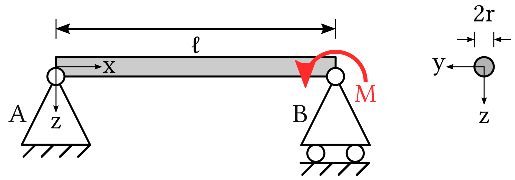
(c) We partition the cross-section into three rectangles $R1$, $R2$, and $R3$ (see above). The contribution of $R1$ can be computed directly using the result from (a). For $R2$ and $R3$ we need the parallel axis theorem. The centers of gravity of both rectangles are $\pm(h+t)/2$ away from the origin. Their contribution is therefore $I_y = \frac{1}{12} t^3 b + l^2 A$, with $l = \pm(h+t)/2$ and $A = tb$. In summary, we have

$$I_y \quad \text{(whole cross-section)} = 2\left(\frac{1}{12} t^3 b + \frac{1}{4}(h+t)^2 tb\right) + \frac{1}{12} dh^3 = \frac{2}{3} t^3 b + ht^2 b + \frac{1}{2} h^2 bt + \frac{1}{12} dh^3.$$

Note that if $d, t \ll b, h$, then $I_y \approx \frac{1}{2} h^2 bt + \frac{1}{12} dh^3$, i.e. the only significant contribution from $R2$ and $R3$ is due to the second term of the parallel axis theorem!

Question 2 ....

A beam with cylindrical cross-section (radius r) is supported by two bearings, see below. A moment M is applied at one end. Calculate the maximum deflection! Where does it occur?

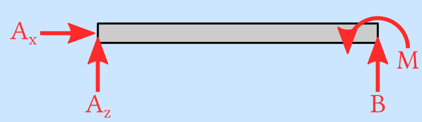


Solution: *Reference*:Gross, Hauger, Schröder, Wall, Technische Mechanik 2, 9th edition, Springer Vieweg (pages 122–123).

Recall that a structure is statically determinate if

$$3n - (r + v) = 0,$$

where n is the number of bodies, r the number of reaction forces or moments of the supports, and v the number of forces or moments transmitted at links. Here $n = 1$, $r = 3$, $v = 0 \rightarrow$ the structure is statically determinate.



From equilibrium, we have $A_x = 0$ and $A_z = -B = -M/l$. The internal moment is

$$M(x) = -xA_z = M\frac{x}{l}.$$

Let E be Young's modulus and I_y the second moment of area for bending about y. Integration of the differential equation of the bending line yields

$$EI_y w'' = -\frac{M}{l} x$$
$$EI_y w' = -\frac{M}{2l} x^2 + C_1$$
$$EI_y w = -\frac{M}{6l} x^3 + C_1 x + C_2$$

The boundary conditions are $w(0) = 0$ and $w(l) = 0$. Inserting into the last equation gives $C_2 = 0$ and $C_1 = \frac{Ml}{6}$. Thus, we have

$$w(x) = \frac{1}{EI_y}\left(-\frac{M}{6l}x^3 + \frac{Ml}{6}x\right).$$

The maximum value of w occurs at the position x^* where $w'(x^*) = 0$, i.e.

$$-\frac{M}{2l}(x^*)^2 + \frac{Ml}{6} = 0 \rightarrow x^* = \frac{1}{\sqrt{3}}l.$$

Thus

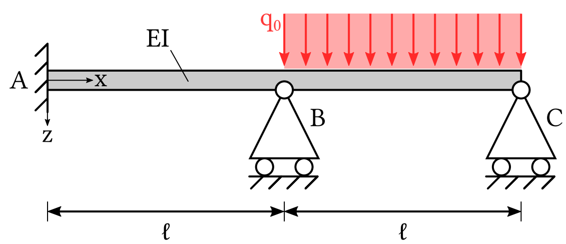
$$w(x^*) = \frac{\sqrt{3}Ml^2}{27EI_y}.$$

For the circular cross-section, we have from exercise 1(b) $I_y = \frac{\pi}{4}r^4$. Inserting gives

$$w(x^*) = \frac{4\sqrt{3}Ml^2}{27\pi Er^4}.$$

Question 3 . . . .

The beam shown below has the bending stiffness EI and is subjected to a line load q_0. Calculate the reaction forces and the deflection of the beam!



Hint: If a system is hyperstatic it might be helpful to start from the Euler-Bernoulli equation before trying to determine the reaction forces.

Solution: This structure is composed of one element ($r = 1$) and four bearings, which create five reactions ($r = 5$). Testing for determinacy, we find

$$3n - (r + v) = -2, \tag{1}$$

i.e. the structure is hyperstatic. We cannot find all reactions by consideration of equilibrium alone. Thus, we will first solve the Euler-Bernoulli equation and then obtain the reactions from the solution.

There is a discontinuity at the support B, hence we need to find separate solutions for the two sectors $0 \leq x \leq \ell$ (sector 1) and $\ell \leq x \leq 2\ell$ (sector 2). Let $w^{(1)}(x)$ be the deflection along z in sector 1. There is no line load, hence

$$EI{w^{(1)}}''''(x) = 0, \tag{2}$$

$$EI{w^{(1)}}'''(x) = C_1^{(1)}, \tag{3}$$

$$EI{w^{(1)}}''(x) = C_1^{(1)}x + C_2^{(1)}, \tag{4}$$

$$EI{w^{(1)}}'(x) = \frac{1}{2}C_1^{(1)}x^2 + C_2^{(1)}x + C_3^{(1)}, \tag{5}$$

$$EIw^{(1)}(x) = \frac{1}{6}C_1^{(1)}x^3 + \frac{1}{2}C_2^{(1)}x^2 + C_3^{(1)}x + C_4^{(1)}, \tag{6}$$

where $C_1^{(1)}$, $C_2^{(1)}$, $C_3^{(1)}$, and $C_4^{(1)}$ are constants of integration.

In sector 2, the line load is q_0, therefore

$$EIw^{(2)''''}(x) = q_0, \tag{7}$$
$$EIw^{(2)'''}(x) = q_0 x + C_1^{(2)}, \tag{8}$$
$$EIw^{(2)''}(x) = \frac{1}{2}q_0 x^2 + C_1^{(2)}x + C_2^{(2)}, \tag{9}$$
$$EIw^{(2)'}(x) = \frac{1}{6}q_0 x^3 + \frac{1}{2}C_1^{(2)}x^2 + C_2^{(2)}x + C_3^{(2)}, \tag{10}$$
$$EIw^{(2)}(x) = \frac{1}{24}q_0 x^4 + \frac{1}{6}C_1^{(2)}x^3 + \frac{1}{2}C_2^{(2)}x^2 + C_3^{(2)}x + C_4^{(2)}, \tag{11}$$

where $C_1^{(2)}$, $C_2^{(2)}$, $C_3^{(2)}$, and $C_4^{(2)}$ are constants of integration.

The following boundary conditions apply:

$$\begin{aligned}
w^{(1)}(x=0) &= 0 \quad \text{(beam is clamped)},\\
w^{(1)}(x=0)' &= 0 \quad \text{(beam is clamped)},\\
w^{(1)}(x=l) &= w^{(2)}(x=l) = 0, \quad \text{(support at } B\text{)},\\
w^{(1)'}(x=l) &= w^{(2)'}(x=l), \quad \text{(no kink at } B\text{)},\\
w^{(1)''}(x=l) &= w^{(2)''}(x=l), \quad \text{(moment continuous at } B\text{)},\\
w^{(2)}(x=2l) &= 0 \quad \text{(support at } C\text{)}.\\
w^{(2)''}(x=2l) &= 0 \quad \text{(no moment at support } C\text{)}.
\end{aligned} \tag{12}$$

By using the first two boundary conditions, we find $C_4^{(1)} = C_3^{(1)} = 0$. Inserting the third boundary condition in Eq. 6, we get

$$C_2^{(1)} = -\frac{1}{3}C_1^{(1)}l. \tag{13}$$

$w^{(2)}(x=l) = 0$ and $w^{(2)}(x=2l) = 0$ imply

$$C_4^{(2)} = -\left(\frac{1}{24}q_0 l^4 + \frac{1}{6}C_1^{(2)}l^3 + \frac{1}{2}C_2^{(2)}l^2 + C_3^{(2)}l\right), \tag{14}$$
$$C_4^{(2)} = -\left(\frac{2}{3}q_0 l^4 + \frac{4}{3}C_1^{(2)} + 2C_2^{(2)}l^2 + 2C_3^{(2)}l\right), \tag{15}$$

which can be combined to give

$$C_3^{(2)} = -\left(\frac{5}{8}q_0 l^3 + \frac{7}{6}C_1^{(2)}l^2 + \frac{3}{2}C_2^{(2)}l\right), \tag{16}$$
$$C_4^{(2)} = \frac{7}{12}q_0 l^4 + C_1^{(2)}l^3 + C_2^{(2)}l^2. \tag{17}$$

$w^{(2)''}(x=2l) = 0$ yields

$$C_2^{(2)} = -2q_0 l^2 - 2C_1^{(2)}l.$$

The only remaining unknowns are now $C_1^{(1)}$ and $C_1^{(2)}$. Thus far, we have

$$EIw^{(1)}(x) = \frac{1}{6}C_1^{(1)}\left(x^3 - lx^2\right),$$
$$EIw^{(2)}(x) = -\frac{1}{24}q_0x^4 + \frac{1}{6}C_1^{(2)}x^3 - \left(q_0l + C_1^{(2)}\right)lx^2 + \frac{19}{8}q_0l^3x + \frac{11}{6}C_1^{(2)}l^2x - \frac{17}{12}q_0l^4 - C_1^{(2)}l^3.$$

Finally, the conditions $w^{(1)\prime}(x=l) = w^{(2)\prime}(x=l)$ and $w^{(1)\prime\prime}(x=l) = w^{(2)\prime\prime}(x=l)$ at B yield

$$C_1^{(1)} = \frac{3}{28}q_0l,$$
$$C_1^{(2)} = -\frac{11}{7}q_0l.$$

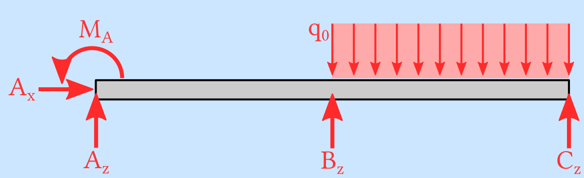
The deflections are therefore

$$w^{(1)}(x) = \frac{1}{EI}\left[\frac{1}{56}q_0l\left(x^3 - lx^2\right)\right],$$
$$w^{(2)}(x) = \frac{1}{EI}\left[-\frac{1}{24}q_0x^4 - \frac{11}{42}q_0lx^3 + \frac{4}{7}q_0l^2x^2 + \frac{19}{8}q_0l^3x - \frac{121}{42}q_0l^3x - \frac{17}{12}q_0l^4 + \frac{11}{7}q_0l^4\right].$$

The reaction forces can be obtained from the derivatives of the deflections,

$$A_z = -EIw^{(1)\prime\prime\prime}(x=0) = -\frac{3}{28}q_0l,$$
$$M_A = -EIw^{(1)\prime\prime}(x=0) = \frac{1}{28}q_0l^2,$$
$$-C_z = -EIw^{(2)\prime\prime\prime}(x=2l) \to C_z = \frac{3}{7}q_0l.$$

$A_x = 0$ and $B_z = \frac{19}{28}q_0l$ follow from equilibrium.



A final note: dividing the structure into different sectors and finding separate solutions, as was done here, can be a bit tedious. A shorter and more elegant solution is possible using MACAULAY brackets (FÖPPEL brackets), see Hauger, Lippmann, Mannl, Werner, Aufgaben zur Technischen Mechanik 1–3, 3d ed. Springer (p. 224–225).