Exercise 8: Stress and strain

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Question 1 . . . .
Reference: Barber, Elasticity, Springer (2010), p. 32

Plastic deformation during a manufacturing process generates a state of stress in the large body $z > 0$. If the stresses are functions of z only and the surface $z = 0$ is not loaded, show that the stress components σ_{yz}, σ_{zx}, σ_{zz} must be zero everywhere!

Solution: In the absence of body forces, the equilibrium condition is

$$\operatorname{div}\sigma = \begin{bmatrix} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} \\ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \end{bmatrix} = 0.$$

Since all stress components are functions of z only, the equilibrium conditions simplifies to

$$\begin{bmatrix} \frac{\partial \sigma_{xz}}{\partial z} \\ \frac{\partial \sigma_{yz}}{\partial z} \\ \frac{\partial \sigma_{zz}}{\partial z} \end{bmatrix} = 0.$$

Therefore σ_{xz}, σ_{yz}, and σ_{zz} are constants. The boundary condition is that the surface is stress-free, i.e. $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$ there. Hence these stresses must be zero everywhere.

Question 2 . . . .
Metal or semiconductor crystals may contain defects in their lattice structure called “dislocations”. These are very important for understanding plastic deformation. A so-called “screw dislocation”, sketched in the figure, is created by the following displacement

$$\mathbf{u}(x, y, z) = \begin{bmatrix} 0 \\ 0 \\ \frac{b}{2\pi}\arctan\left(\frac{y}{x}\right) \end{bmatrix}.$$

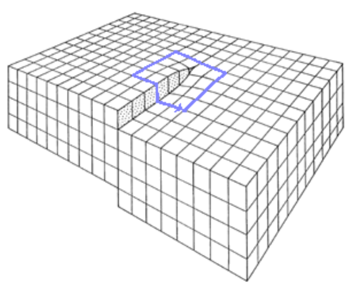

Figure 1: screw dislocation from: https://www.tf.uni -kiel.de/matwis/amat/def_en/kap_5/backbone/r5_2_2.html

Calculate the associated strain tensor ε and the stress tensor σ (using Hooke’s law)! Is the body in a state of plane strain or plane stress? Do you notice something peculiar near the center of the dislocation at $x = y = 0$?

Solution: The strains are given by the equation

$$\varepsilon_{ij} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)$$

we thus find

$$\begin{aligned}
\varepsilon_{xx} &= \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = \varepsilon_{yx} = 0, \\
\varepsilon_{xz} &= \varepsilon_{zx} = -\frac{b}{4\pi}\frac{y}{x^2+y^2}, \\
\varepsilon_{yz} &= \varepsilon_{zy} = \frac{b}{4\pi}\frac{x}{x^2+y^2},
\end{aligned}$$

and the stresses can be computed by the formula for isotropic materials given in the lecture

$$\sigma_{ij} = \lambda \delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$$

to find

$$\begin{aligned}
\sigma_{xx} &= \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yx} = 0, \\
\sigma_{xz} &= \sigma_{zx} = -\frac{\mu b}{2\pi}\frac{y}{x^2+y^2}, \\
\sigma_{yz} &= \sigma_{zy} = \frac{\mu b}{2\pi}\frac{x}{x^2+y^2}.
\end{aligned}$$

The state of deformation is neither plane strain nor plane stress. Note that the fields diverge as $x, y \to 0$. Thus small strain elasticity breaks down in some region around $x = y = 0$ and one needs to consider the atomic structure of the material to find the true state of deformation.

Question 3 . . . .

We now consider a state of plane strain. The governing equations are

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \quad \text{(definition of strain)},$$

$$\sigma_{xx} = 2\mu\varepsilon_{xx} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy}\right), \quad \sigma_{yy} = 2\mu\varepsilon_{yy} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy}\right), \quad \sigma_{xy} = 2\mu\varepsilon_{xy} \quad \text{(Hooke's law)},$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + F_x = 0, \quad \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} + F_y = 0, \quad \text{(equilibrium)}.$$

These are eight governing equations. However, we can combine them in such a way that we end up with only two equations in terms of the displacement components u_x and u_y. This form is convenient for problems where displacement components are prescribed over the entire boundary of the body. Find these two equations!

Solution: By subsituting the strains into Hooke's law, one obtains

$$\begin{aligned}
\sigma_{xx} &= \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + 2\mu\frac{\partial u_x}{\partial x}, \\
\sigma_{yy} &= \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + 2\mu\frac{\partial u_y}{\partial y}, \\
\sigma_{xy} &= \mu\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right).
\end{aligned}$$

Inserting these equations into the equilibrium conditions and eliminating stresses gives

$$\begin{aligned}
\mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right) + (\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + F_x &= 0, \\
\mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right) + (\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + F_y &= 0.
\end{aligned}$$

These are the *Navier-Cauchy* equations for plane strain.

Question 4

We want to demonstrate for the two-dimensional case that Hooke's law with isotropic elastic constants is indeed isotropic. Consider a 2D stress tensor σ and the corresponding strain ε,

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{bmatrix}.$$

Next, consider the matrix for rotation by an arbitrary angle α

$$R = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}.$$

The most straightforward way to demonstrate isotropy would be to rotate the elastic stiffness tensor. However, this is a fourth-order tensor and rotating it is cumbersome. Here, we take a different approach. In order to demonstrate isotropy

1. express σ in terms of the components of ε,
2. rotate σ to find the representation σ' of this state of stress in the new coordinate system,
3. replace the components of ε in σ' by the components of the strain tensor ε' in the rotated coordinate system.

You should see that the constants of proportionality between stress and strain — the elastic constants — are the same in the new and the old coordinate system!

Solution:

1.)

$$\begin{aligned} \sigma_{ij} &= \lambda \delta_{ij} \varepsilon_{kk} + 2\mu \varepsilon_{ij} \\ \underline{\sigma} &= \begin{bmatrix} 2\mu\varepsilon_{xx} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy}\right) & 2\mu\varepsilon_{xy} \\ 2\mu\varepsilon_{xy} & 2\mu\varepsilon_{yy} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy}\right) \end{bmatrix} \end{aligned}$$

2.) Assuming that R is the matrix which, given the representation of a vector in the original coordinate system, yields the representation in the new coordinate system, we need to perform the following operation to find σ':

$$\begin{aligned} \sigma' &= R\sigma R^T \quad \text{(matrix notation), or, equivalently,} \\ \sigma'_{mn} &= R_{mi} R_{nj} \sigma_{ij} \quad \text{(index notation).} \end{aligned}$$

The result is

$$\begin{aligned} \sigma' &= \begin{bmatrix} \cos(\alpha)^2\sigma_{xx} + \sin(2\alpha)\sigma_{xy} + \sin(\alpha)^2\sigma_{yy} & \cos(2\alpha)\sigma_{xy} - \cos(\alpha)\sin(\alpha)\left(\sigma_{xx} - \sigma_{yy}\right) \\ \cos(2\alpha)\sigma_{xy} - \cos(\alpha)\sin(\alpha)\left(\sigma_{xx} - \sigma_{yy}\right) & \sin(\alpha)^2\sigma_{xx} - 2\sin(\alpha)\cos(\alpha)\sigma_{xy} + \cos(\alpha)^2\sigma_{yy} \end{bmatrix} \\ &= \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} \\ \sigma'_{xy} & \sigma'_{yy} \end{bmatrix}, \quad \text{with} \\ \sigma'_{xx} &= \left(\varepsilon_{xx} + \varepsilon_{yy}\right)(\mu + \lambda) + \left(\varepsilon_{xx} - \varepsilon_{yy}\right)\mu\cos(2\alpha) + 2\varepsilon_{xy}\mu\sin(2\alpha), \\ \sigma'_{yy} &= \left(\varepsilon_{xx} + \varepsilon_{yy}\right)(\mu + \lambda) - \left(\varepsilon_{xx} - \varepsilon_{yy}\right)\mu\cos(2\alpha) - 2\varepsilon_{xy}\mu\sin(2\alpha), \\ \sigma'_{xy} &= \mu\left(2\varepsilon_{xy}\cos(2\alpha) - \left(\varepsilon_{xx} - \varepsilon_{yy}\right)\sin(2\alpha)\right). \end{aligned}$$

3.) To get the components of ε in terms of the components of ε', we need to consider the reverse sense of rotation, i.e. $\varepsilon = R^T \varepsilon' R$. The transformation rules are the same for stress and strain, therefore the result can be obtained immediately by replacing $\alpha \to -\alpha$, $\sigma_{xx} \to \varepsilon'_{xx}$, $\sigma_{xy} \to \varepsilon'_{xy}$, and $\sigma_{yy} \to \varepsilon'_{yy}$ in the matrix above,

$$\varepsilon = \begin{bmatrix} \cos(\alpha)^2\varepsilon'_{xx} - \sin(2\alpha)\varepsilon'_{xy} + \sin(\alpha)^2\varepsilon'_{yy} & \cos(2\alpha)\varepsilon'_{xy} + \cos(\alpha)\sin(\alpha)\left(\varepsilon'_{xx} - \varepsilon'_{yy}\right) \\ \cos(2\alpha)\varepsilon'_{xy} + \cos(\alpha)\sin(\alpha)\left(\varepsilon'_{xx} - \varepsilon'_{yy}\right) & \sin(\alpha)^2\varepsilon'_{xx} + 2\sin(\alpha)\cos(\alpha)\varepsilon'_{xy} + \cos(\alpha)^2\varepsilon'_{yy} \end{bmatrix}.$$

Inserting the components of ε in the equations for σ'_{xx}, σ'_{xy}, and σ'_{yy}, we obtain

$$\sigma' = \begin{bmatrix} 2\mu\varepsilon'_{xx} + \lambda\left(\varepsilon'_{xx} + \varepsilon'_{yy}\right) & 2\mu\varepsilon'_{xy} \\ 2\mu\varepsilon'_{xy} & 2\mu\varepsilon'_{yy} + \lambda\left(\varepsilon'_{xx} + \varepsilon'_{yy}\right) \end{bmatrix}.$$

We can see that the elastic constants are the same in the two coordinate systems.