Exercise 7: Strain
27.11.2023 - 01.12.2023

Question 1 . . . .

Consider the following displacement field,

$$\mathbf{u}(x, y, z) = k \begin{bmatrix} 2x + y^2 \\ x^2 - 3y^2 \\ 0 \end{bmatrix},$$

where k is a nonzero constant. Calculate the strain tensor ε!

Solution: The strain tensor ε was defined in the lecture by

$$\varepsilon = \frac{1}{2}\left(\vec{\nabla}\vec{u} + \left(\vec{\nabla}\vec{u}\right)^T\right) \quad \text{or in index notation} \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right)$$

With this formulas you are able to compute each component of the strain tensor

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial x}\right) = \frac{\partial(2x + y^2)}{\partial x} = 2k \\
\varepsilon_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial(2x + y^2)}{\partial y} + \frac{\partial(x^2 - 3y^2)}{\partial x}\right) = \frac{1}{2}(2ky + 2kx) = k(x + y) \\
\varepsilon_{yy} &= \frac{1}{2}\left(\frac{\partial u_y}{\partial y} + \frac{\partial u_y}{\partial y}\right) = \frac{\partial(x^2 - 3y^2)}{\partial y} = -6ky \\
&\ldots
\end{aligned}$$

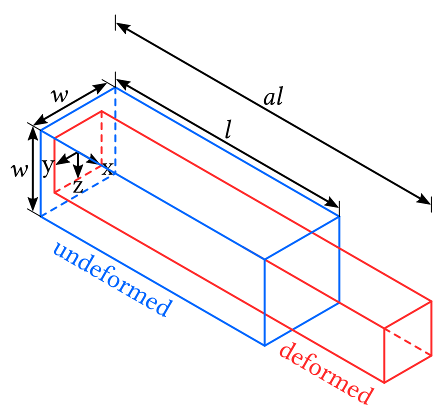
By using the symmetry of ε you only have to compute 6 entries and should find the following result

$$\varepsilon = k \begin{bmatrix} 2 & x + y & 0 \\ x + y & -6y & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that all components of ε involving the z-direction are zero. This situation is called *plane strain*.

Question 2 . . . .

A solid bar with dimensions $l \times w \times w$ (see below) is stretched along its length to a final length al. The volume of the bar does not change during deformation. Calculate the displacement vector $\mathbf{u}$ and the strain tensor ε!



Solution: After stretching, the bar has a new width $\hat{w}$. However, the volume is conserved, therefore

$$\hat{w}^2 la = w^2 l \implies \hat{w} = \frac{w}{\sqrt{a}}.$$

We assume the bar deforms homogeneously. The displacement along the x-direction increases linearly from zero at $x = 0$ to $(a-1)l$ at $x = l$. Similarly, the displacement in y-direction increases linearly from zero at $y = 0$ to some maximum value at $y = w/2$. The displacement in z-direction increases linearly from zero at $z = 0$ to some maximum value at $z = w/2$. The bar retains its square cross-section, i.e. the y-displacement does not depend on x and z, and the z-displacement does not depend on y and x. Therefore, the displacement vector can be written as

$$\mathbf{u} = \begin{bmatrix} (a-1)x \\ Ay + B \\ Cz + D \end{bmatrix},$$

where A, B, C, D are constants that need to be determined by consideration of the boundary conditions. Since the y- and z- components are zero at $y = 0$ and $z = 0$, respectively, we see that $B = D = 0$. The bar retains its square cross-section, i.e. the y-displacement at $y = w/2$ must be equal to the z-displacement at $z = w/2$. Therefore $A = C$. Recall that the new width after deformation is $\hat{w} = w/\sqrt{a}$. For this reason, the y-displacement at $y = w/2$ must be equal to $(\hat{w} - w)/2 = (1/\sqrt{a} - 1)w/2$. It must also be equal to $Aw/2$. Therefore $A = (1/\sqrt{a} - 1)$. In conclusion, the displacement vector is

$$\mathbf{u} = \begin{bmatrix} (a-1)x \\ \left(\frac{1}{\sqrt{a}} - 1\right) y \\ \left(\frac{1}{\sqrt{a}} - 1\right) z \end{bmatrix}.$$

Through differentiation, we obtain the strain tensor

$$\varepsilon = \begin{bmatrix} a-1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{a}} - 1 & 0 \\ 0 & 0 & \frac{1}{\sqrt{a}} - 1 \end{bmatrix}.$$

Question 3 . . . .
(Saint-Venant's compatibility conditions)

The strain tensor ε has six distinct components. However, these six components are computed from only three components of the displacement vector $\mathbf{u}$. Thus, if we want to solve for the components of $\mathbf{u}$ *given* the component of ε, we have six equations for three unknowns. For this system of equations to have a solution, some of the strain components must be related. Show that they are by considering their second derivatives! For example, differentiate ε_{xx} twice with respect to y, ε_{yy} twice with respect to x and ε_{xy} with respect to x and y, and compare!

Solution: Recall the definition of the aforementioned strain components,

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x},$$
$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y},$$
$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right).$$

Thus,

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial^3 u_x}{\partial x \partial y^2},$$
$$\frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^3 u_y}{\partial y \partial x^2},$$

and

$$\begin{aligned}\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} &= \frac{1}{2}\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right),\\ &= \frac{1}{2}\left(\frac{\partial^3 u_x}{\partial y^2 \partial x} + \frac{\partial^3 u_y}{\partial x^2 \partial y}\right).\end{aligned}$$

The order of differentiation in the last equation is immaterial, hence we see that

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}.$$

Two more equations of this form can be obtained by repeating this procedure for ε_{yy}, ε_{zz}, and ε_{yz}, and for ε_{xx}, ε_{zz}, and ε_{xz}. This is tantamount to cyclic permutations of the indices, $x \to y$, $y \to z$, and $z \to x$.

Three more equations can be obtained by considering mixed derivatives of type

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial^3 u_x}{\partial x \partial y \partial z}.$$

Again, order of differentiation is immaterial, hence

$$\begin{aligned}\frac{\partial^3 u_x}{\partial x \partial y \partial z} &= \frac{\partial^2}{\partial x \partial z}\left(\frac{\partial u_x}{\partial y}\right)\\ &= \frac{\partial^2}{\partial x \partial z}\left(2\varepsilon_{xy} - \frac{\partial u_y}{\partial x}\right)\\ &= \frac{\partial}{\partial x}\left(2\frac{\partial \varepsilon_{xy}}{\partial z} - \frac{\partial^2 u_y}{\partial x \partial z}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial^3 u_x}{\partial x \partial y \partial z} &= \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u_x}{\partial z}\right)\\ &= \frac{\partial^2}{\partial x \partial y}\left(2\varepsilon_{xz} - \frac{\partial u_z}{\partial x}\right)\\ &= \frac{\partial}{\partial x}\left(2\frac{\partial \varepsilon_{xz}}{\partial y} - \frac{\partial^2 u_z}{\partial x \partial y}\right).\end{aligned}$$

Combing gives

$$\begin{aligned}2\frac{\partial^3 u_x}{\partial x \partial y \partial z} &= \frac{\partial}{\partial x}\left(2\frac{\partial \varepsilon_{xy}}{\partial z} - \frac{\partial^2 u_y}{\partial x \partial z}\right) + \frac{\partial}{\partial x}\left(2\frac{\partial \varepsilon_{xz}}{\partial y} - \frac{\partial^2 u_z}{\partial x \partial y}\right)\\ &= \frac{\partial}{\partial x}\left(2\frac{\partial \varepsilon_{xy}}{\partial z} + 2\frac{\partial \varepsilon_{xz}}{\partial y} - \frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y}\right)\\ &= 2\frac{\partial}{\partial x}\left(\frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{xz}}{\partial y} - \frac{\partial \varepsilon_{yz}}{\partial x}\right).\end{aligned}$$

Finally,

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{xz}}{\partial y} - \frac{\partial \varepsilon_{yz}}{\partial x}\right)$$

The other two equations of this type can be obtained by cyclic permutation of the indices, $x \to y$, $y \to z$, and $z \to x$. The six compatibility conditions are therefore

$$\begin{aligned}
2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} &= \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} \quad \text{(a)},\\
2\frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} &= \frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} \quad \text{(b)},\\
2\frac{\partial^2 \varepsilon_{zx}}{\partial z \partial x} &= \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} \quad \text{(c)},\\
\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} &= \frac{\partial}{\partial x}\left(\frac{\partial \varepsilon_{xy}}{\partial z} + \frac{\partial \varepsilon_{xz}}{\partial y} - \frac{\partial \varepsilon_{yz}}{\partial x}\right) \quad \text{(d)},\\
\frac{\partial^2 \varepsilon_{yy}}{\partial z \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial \varepsilon_{yz}}{\partial x} + \frac{\partial \varepsilon_{yx}}{\partial z} - \frac{\partial \varepsilon_{zx}}{\partial y}\right) \quad \text{(e)},\\
\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} &= \frac{\partial}{\partial z}\left(\frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{zy}}{\partial x} - \frac{\partial \varepsilon_{xy}}{\partial z}\right) \quad \text{(f)}.
\end{aligned}$$

Question 4 . . . .

In the first question, you computed the strain tensor ε for displacement field

$$\mathbf{u}(x,y,z) = k\begin{bmatrix} 2x + y^2 \\ x^2 - 3y^2 \\ 0 \end{bmatrix}.$$

Now show that ε fulfills the compatibility conditions!

Solution: Recall that

$$\varepsilon = kH\begin{bmatrix} 2 & x+y & 0 \\ x+y & -6y & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that all components of ε involving the z-direction are zero (*plane strain*). All derivatives with respect to z are zero. Moreover, ε is linear in x and y. Therefore, all terms of the type $\partial^2(\ldots)/\partial x^2$ and $\partial^2(\ldots)/\partial y^2$ are zero. The only non zero term that is left over is $\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}$. We can inspect this term in the first condition (a)

$$\begin{aligned}
& & 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} &= 0 + 0\\
&\Leftrightarrow & 2\frac{\partial}{\partial y}\left(\frac{\partial \varepsilon_{xy}}{\partial x}\right) &= 0\\
&\Leftrightarrow & 2\frac{\partial}{\partial y}(1) &= 0\\
&\Leftrightarrow & 0 &= 0 \quad \text{OK!}
\end{aligned}$$

so the derivative $\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = 0$. Thus, we can immediately see that conditions (b)–(f) are fulfilled. Therefore the strains are compatible.