Exercise 4: Divergence and static equilibrium

13.11.2023 - 17.11.2023

Question 1 . . . .

Reference: Chou, Pagano, *Elasticity: Tensor, Dyadic, and Engineering Approaches*, Dover Publications, pages 32–33.

(a) Is the following stress distribution possible for a body in equilibrium? A, B, and C are constants. Body forces are zero.

$$\begin{aligned}
\sigma_{xx} &= -Axy \\
\tau_{xy} &= \frac{A}{2}\left(B^2 - y^2\right) + Cz \\
\tau_{xz} &= -Cy \\
\sigma_{yy} &= \sigma_{zz} = \tau_{yz} = 0
\end{aligned}$$

(b) Check whether equilibrium exists for the following stress distribution. Body forces are zero.

$$\begin{aligned}
\sigma_{xx} &= 3x^2 + 4xy - 8y^2 \\
\sigma_{yy} &= 2x^2 + xy + 3y^2 \\
\tau_{xy} &= -\frac{1}{2}x^2 - 6xy - 2y^2 \\
\sigma_{zz} &= \sigma_{zx} = \tau_{yz} = 0
\end{aligned}$$

Solution: Recall that

$$\operatorname{div} \boldsymbol{\sigma} = \boldsymbol{\sigma} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} + \frac{\partial \sigma_{13}}{\partial z} \\ \frac{\partial \sigma_{21}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y} + \frac{\partial \sigma_{23}}{\partial z} \\ \frac{\partial \sigma_{31}}{\partial x} + \frac{\partial \sigma_{32}}{\partial y} + \frac{\partial \sigma_{33}}{\partial z} \end{bmatrix}.$$

For (a) we get

$$\operatorname{div} \boldsymbol{\sigma} = \begin{bmatrix} -2Ay \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, equilibrium exists only if A is zero.

For (b) we get

$$\operatorname{div} \boldsymbol{\sigma} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore equilibrium exists.

Question 2 . . . .
You might have some problems in computing and getting a feeling of a divergence. In the lecture you have learned about the divergence theorem, also called the Gauss theorem, which can help you in understanding the divergence of a vector field.

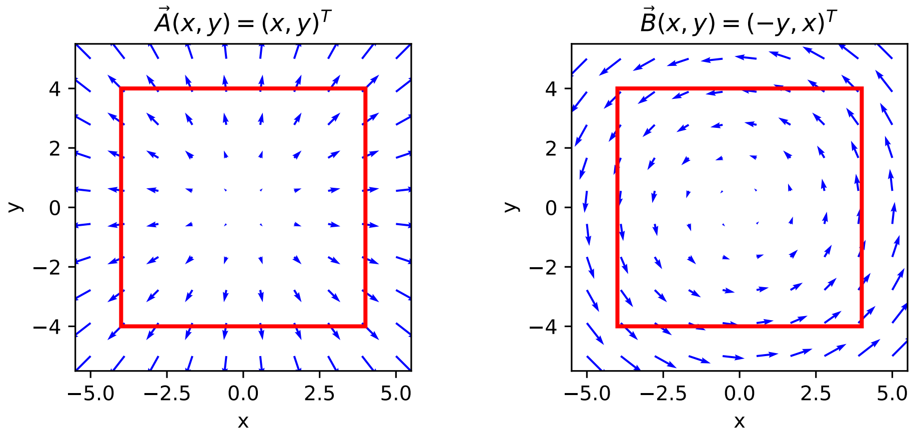


Figure 1: Two different vector fields $\vec{A}$ and $\vec{B}$, indicated by arrows in blue. In red the surface $\partial\Omega$ enclosing the area Ω used for the divergence theorem.

(a) Let us start in computing the divergence of the two vector fields sketch in figure 1. The first vector field $\vec{A}(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}$ could be the velocity of water coming out of the tip from a cone and running down at the surface. The second one, $B(x,y) = \begin{pmatrix} -y \\ x \end{pmatrix}$, could be the velocity of particles fixed on a rotating plate.

(b) Now you can have a look at the divergence theorem

$$\int_\Omega \vec{\nabla} \cdot \vec{F} \mathrm{d}\Omega = \int_{\partial\Omega} \vec{n} \cdot \vec{F} \mathrm{d}\partial\Omega$$

for a vector field $\vec{F}$ and the area Ω enclosed by the surface $\partial\Omega$ with surface normal vector $\vec{n}$ pointing outwards. Compute the left **and** the right hand side of the divergence theorem for the vector fields $\vec{A}$ and $\vec{B}$ form part (a). The area Ω is given by the red box in the figure. To compute the right hand side of the divergence theorem it is useful to subdivide the integral into four parts, the four sides of the square.

(c) Now try to interpret your findings from (a) and (b). You can therefore imagine squares of arbitrary size at different positions in the two plots of the vector fields. How does the vector field behave/change inside the square areas? Are there "more" vectors pointing out of the area than coming in? Is there a relation to the divergence computed in (a)? Can you describe the quantities computed by the left and the right side of the divergence theorem in (b) pictorially in the figures?

Solution:

(a) For vector field $\vec{A}$: $\mathrm{div}(A) = \vec{\nabla} \cdot \vec{A} = \partial_x A_x + \partial_y A_y = 1 + 1 = 2$
For vector field $\vec{B}$: $\mathrm{div}(B) = \vec{\nabla} \cdot \vec{B} = \partial_x B_x + \partial_y B_y = 0 + 0 = 0$

(b) The area Ω are the points enclosed by the red square in figure 1. Thus $\Omega = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \text{ for } x, y \in [-4, 4] \right\}$
We can now compute both sides of the divergence theorem.

For $\vec{A}$:

$$\int_\Omega \vec{\nabla}\cdot\vec{A}\mathrm{d}\Omega = \int_\Omega 2\,\mathrm{d}\Omega = \int_{-4}^{4}\int_{-4}^{4} 2\,\mathrm{d}x\mathrm{d}y = 128$$

The surface integral can be subdivided in four integrals along the edges of the red square. The surface normals are given by the unit vectors in $\pm\vec{e}_x = \begin{pmatrix}\pm1\\0\end{pmatrix}$ and $\pm\vec{e}_y = \begin{pmatrix}0\\\pm1\end{pmatrix}$. Along one line of the square either x or y has a constant value which can be replaced in the vector field.

$$\begin{aligned}\int_{\partial\Omega}\vec{n}\cdot\vec{A}\mathrm{d}(\partial\Omega) &= \int_{-4}^{4}\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}x\\4\end{pmatrix}\mathrm{d}x + \int_{-4}^{4}\begin{pmatrix}-1\\0\end{pmatrix}\cdot\begin{pmatrix}-4\\y\end{pmatrix}\mathrm{d}y + \int_{-4}^{4}\begin{pmatrix}0\\-1\end{pmatrix}\cdot\begin{pmatrix}x\\-4\end{pmatrix}\mathrm{d}x + \int_{-4}^{4}\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}4\\y\end{pmatrix}\mathrm{d}y\\ &= \int_{-4}^{4}4\,\mathrm{d}x + \int_{-4}^{4}4\,\mathrm{d}y + \int_{-4}^{4}4\,\mathrm{d}x + \int_{-4}^{4}4\,\mathrm{d}y = 32+32+32+32 = 128\end{aligned}$$

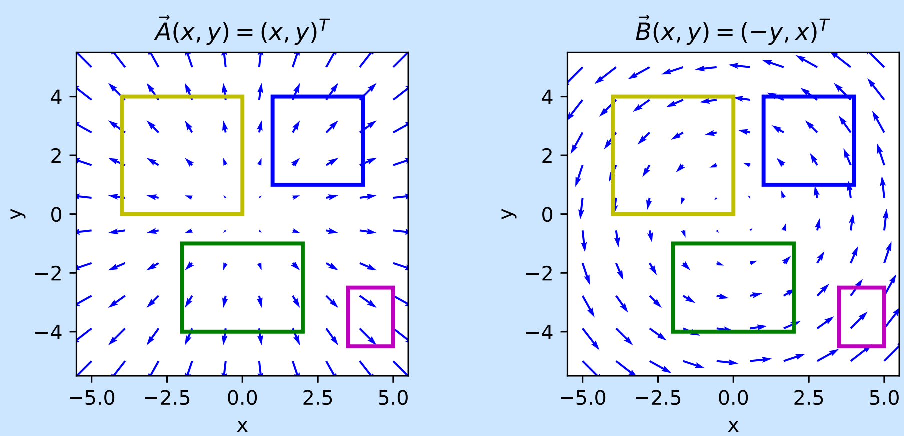
For $\vec{B}$:

$$\int_\Omega \vec{\nabla}\cdot\vec{B}\mathrm{d}\Omega = \int_\Omega 0\,\mathrm{d}\Omega = 0$$

We subdivide the surface integral as for vector field $\vec{A}$.

$$\begin{aligned}\int_{\partial\Omega}\vec{n}\cdot\vec{B}\mathrm{d}(\partial\Omega) &= \int_{-4}^{4}\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}-4\\x\end{pmatrix}\mathrm{d}x + \int_{-4}^{4}\begin{pmatrix}-1\\0\end{pmatrix}\cdot\begin{pmatrix}-y\\4\end{pmatrix}\mathrm{d}y + \int_{-4}^{4}\begin{pmatrix}0\\-1\end{pmatrix}\cdot\begin{pmatrix}4\\x\end{pmatrix}\mathrm{d}x + \int_{-4}^{4}\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}-y\\4\end{pmatrix}\mathrm{d}y\\ &= \int_{-4}^{4}x\,\mathrm{d}x + \int_{-4}^{4}y\,\mathrm{d}y + \int_{-4}^{4}-x\,\mathrm{d}x + \int_{-4}^{4}-y\,\mathrm{d}y = 0+0+0+0 = 0\end{aligned}$$

(c) Now we try to understand the computed values from part (a) and (b). To understand quantities computed in the divergence theorem we can imagine squares at different positions in the two vector fields as indicated in the figure. As the divergence is constant (this is usually not the case and was only chosen here to make it simpler) and even zero for vector field $\vec{B}$, the right hand side of the divergence theorem is easy to compute.



For vector field $\vec{A}$ you can realize that for each square the field vectors are pointing in average outwards. If the field is pointing in on one side of the square it is pointing out stronger on the other sides (e.g. green square). So in average the field is increasing over such an area. One could say there is a **source** of that field. Thinking in the picture of water flowing down the surface of a cone, the water is accelerated by the downhill force which is everywhere equivalent on the cone surface. This is what we have computed, $\mathrm{div}(\vec{A})$ is everywhere constant and positive. By the surface integral you count the part of the field which is crossing the surface. If you do this for a closed surface you find the change of the vector field, which for this example is always positive.

For vector field $\vec{B}$ you can do an analogous analysis. The divergence of the field is zero, which means there is no source which would increase the amplitude of the field. You can observe this also in the surface integral, arrows pointing outwards at one side are pointing inwards somewhere else so that in average it always sums up to zero. Think about the physical analogy of the vector field $\vec{B}$ as the velocity of particles fixed on a plate which is rotating. Then it is clear that te particles are not accelerated or decelerated, they are moving with constant velocity in circles, so there is no source which is changing the absolute velocity.

This image of the divergence as a source for the field is often useful. For example in electrostatics you can find the electric charge as the source of the electrostatic field, $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ or also in terms of the right hand side of the divergence theorem $\oint_S \vec{E} \cdot \mathrm{d}\vec{A} = Q_{\text{enclosed by S}}/\epsilon_0$. However this picture might also mislead you, so double check if your picture corresponds with your computations and vice versa.